# Basic idea of Modelling with Fractional Differential Equations 

Andrew Omame, Ph.D

Abdus Salam School of Mathematical Sciences GC University Lahore, Pakistan. (Presentation at Kong Research Lab Meeting)

April 28, 2023

The major highlights of this presentation are as follows:
i. A brief introduction to fractional calculus and some methodologies.
ii. Basic idea of the Non-standard finite difference scheme
iii. Solution of a simple fractional-order single population growth model

## Introduction

- In applied mathematics and mathematical analysis, fractional derivative is a derivative of any arbitrary order, real or complex.
- The concept of a fractional derivative was coined by the famous mathematician Leibnitz in 1695 in his letter to L'Hopital.
- The new theory has turned out to be very attractive to mathematicians as well as physicists, biologists, engineers, and economists.
- In recent years, the Fractional Calculus (FC) draws increasing attention due to its applications in many fields.
- Various types of fractional derivatives have been studied: Riemann-Liouville, Caputo, Hadamard, Erdelyi-Kober, Grunwald-Letnikov, Marchaud, and Riesz, Caputo-Fabrizio, Atangana-Baleanu, just to mention few.
- The fractional derivatives have their advantages and disadvantages.
- In fractional calculus, the fractional derivatives are defined via fractional integrals.
- The real physical interpretation of fractional derivative in real life problems is still an open problem


## Advantages of the fractional derivatives

- The kernel function of fractional derivative is called memory function, although it does not reflect any physical process. Classical integer-order ordinary differential equations have no memory, because their solution does not depend on the previous instant.
- One of the great advantages of the Caputo fractional derivative is that it allows initial and boundary conditions to be included in the formulation of the problem.
- In addition, its derivative for a constant is zero.
- The Caputo derivative is the most appropriate fractional operator to be used in modeling real world problem.
- The Atangana-Baleanu fractional derivative is more generalized and very appropriate in modeling the behavior of orthodox viscoelastic materials, thermal medium, etc.


## Disadvantages of the fractional derivatives

- The Riemann-Liouville derivative of a constant is not zero.
- In addition, if an arbitrary function is a constant at the origin, its fractional derivation has a singularity at the origin, for instance, exponential and Mittag-Leffler functions. These disadvantages reduce the field of application of the Riemann-Liouville fractional derivative.
- Caputo's derivative demands higher conditions of regularity for differentiability: to compute the fractional derivative of a function in the Caputo sense, we must first calculate its derivative.
- Caputo derivatives are defined only for differentiable functions while functions that have no first-order derivative might have fractional derivatives of all orders less than one in the Riemann-Liouville sense.
- With the Caputo-Fabrizio derivative, the kernel is local and its derivative when $\alpha=0$ does not give the initial function. In addition to this, the anti-derivative associated with the Caputo-Fabrizio operator is not fractional.
- The Atangana-Baleanu derivative has the problem of initial conditions.


## Preliminaries and methodologies

Let us consider a continuous function $y=f(t)$. According to the well-known definition, the first-order derivative of the function $f(t)$ is defined by

$$
\begin{equation*}
f^{\prime}(t)=\frac{d f}{d t}=\lim _{h \rightarrow 0} \frac{f(t)-f(t-h)}{h} \tag{2.1}
\end{equation*}
$$

Applying this definition twice gives the second-order derivative:

$$
\begin{align*}
f^{\prime \prime}(t)=\frac{d^{2} f}{d t^{2}} & =\lim _{h \rightarrow 0} \frac{f^{\prime}(t)-f^{\prime}(t-h)}{h} \\
& =\lim _{h \rightarrow 0} \frac{1}{h}\left\{\frac{f(t)-f(t-h)}{h}-\frac{f(t-h)-f(t-2 h)}{h}\right\}  \tag{2.2}\\
& =\lim _{h \rightarrow 0} \frac{f(t)-2 f(t-h)+f(t-2 h)}{h^{2}}
\end{align*}
$$

## Preliminaries and methodologies

Using (2.1) and (2.2), we obtain

$$
\begin{equation*}
f^{\prime \prime \prime}(t)=\frac{d^{3} f}{d t^{3}}=\lim _{h \rightarrow 0} \frac{f(t)-3 f(t-h)+3 f(t-2 h)-f(t-3 h)}{h^{3}} \tag{2.3}
\end{equation*}
$$

and, by induction,

$$
\begin{equation*}
f^{(n)}(t)=\frac{d^{n} f}{d t^{n}}=\lim _{h \rightarrow 0} \frac{1}{h^{n}} \sum_{r=0}^{n}(-1)^{r}\binom{n}{r} f(t-r h) \tag{2.4}
\end{equation*}
$$

where

$$
\begin{equation*}
\binom{n}{r}=\frac{n(n-1)(n-2) \ldots(n-r+1)}{r!} \tag{2.5}
\end{equation*}
$$

is the usual notation for the binomial coefficients.

## Basic concepts on Laplace transform

The Laplace transform of the function $f(t)$ is defined as:

$$
\begin{equation*}
F(s)=L\{f(t) ; s\}=\int_{0}^{\infty} e^{-s t} f(t) d t, \quad s>0 \tag{2.6}
\end{equation*}
$$

For the existence of the integral (2.6), the function $f(t)$ must be of exponential order $\alpha$, which means that there exist positive constants $M$ and $T$ such that $e^{-\alpha t}|f(t)| \leq M$ for all $t>T$. The convolution of two functions $f$ and $g$ is defined as

$$
\begin{equation*}
f(t) * g(t)=\int_{0}^{t} f(t-\tau) g(\tau) d \tau=\int_{0}^{t} f(\tau) g(t-\tau) d \tau \tag{2.7}
\end{equation*}
$$

The Laplace transform of the convolution of $f$ and $g$ is defined as:

$$
\begin{equation*}
\mathcal{L}\{f(t) * g(t)\}=F(s) G(s) \tag{2.8}
\end{equation*}
$$

## Definitions of some fractional derivatives

## Definition 1

[2] Let $f \in H^{1}(a, b), \alpha \in(0,1)$, where $H^{1}(a, b)=\left\{f \in L^{2}(a, b): D f \in L^{2}(a, b)\right\}$. Then the Riemann-Liouville fractional derivative of the function $f$ of order $\alpha$ is defined by

$$
\begin{equation*}
{ }_{a}^{R L} D_{t}^{\alpha} f(t)=\frac{1}{\Gamma(1-\alpha)} \frac{d}{d t} \int_{a}^{t}(t-\tau)^{-\alpha} f(\tau) d \tau \tag{2.9}
\end{equation*}
$$

The symbol 「 stands for the Gamma function defined by

$$
\begin{equation*}
\Gamma(\alpha)=\int_{a}^{\infty} \exp (-\tau) \tau^{\alpha-1} d \tau, \quad \Gamma(\alpha+1)=\alpha!=\alpha \Gamma(\alpha) \tag{2.10}
\end{equation*}
$$

## Definition 2

[1] The Laplace transform of the Riemann-Liouville fractional derivative is given by

$$
\begin{equation*}
\mathcal{L}\left\{{ }_{a}^{R L} D_{t}^{\alpha} f(t)\right\}(s)=s^{\alpha} \mathcal{L}\{f(t)\}-\left.{ }_{a}^{R L} D_{t}^{\alpha} f(t)\right|_{t=a}, \quad 0<\alpha<1 \tag{2.11}
\end{equation*}
$$

where $\mathcal{L}$ is the Laplace transform operator. It is also assumed that $f$ is a piecewise-continuous function of exponential order.

## Definition 3

[2] Let $f \in H^{1}(a, b), \alpha \in(0,1)$, where $H^{1}(a, b)=\left\{f \in L^{2}(a, b): D f \in L^{2}(a, b)\right\}$. Then the Caputo fractional derivative of the function $f$ of order $\alpha$ is defined by

$$
\begin{equation*}
{ }_{a}^{c} D_{t}^{\alpha} f(t)=\frac{1}{\Gamma(1-\alpha)} \int_{a}^{t}(t-\tau)^{-\alpha} f^{\prime}(\tau) d \tau \tag{2.12}
\end{equation*}
$$

## Definition 4

[2] The Caputo fractional integral of a function $f$ of order $\alpha$ is defined by

$$
\begin{equation*}
{ }_{a}^{c} l_{t}^{\alpha} f(t)=\frac{1}{\Gamma(\alpha)} \int_{a}^{t}(t-\tau)^{\alpha-1} f(\tau) d \tau, \quad t>0 \tag{2.13}
\end{equation*}
$$

If $f(t)=1$, the fractional integral of order $\alpha>0$ is given by

$$
\begin{equation*}
{ }_{a}^{C} I_{t}^{\alpha}(1)=\frac{1}{\Gamma(\alpha)} \int_{a}^{t}(t-\tau)^{\alpha-1}(1) d \tau=\frac{t^{\alpha}}{\Gamma(\alpha+1)} . \tag{2.14}
\end{equation*}
$$

## Definition 5

[2] The Laplace transform of Caputo fractional derivative is given by

$$
\begin{equation*}
\mathcal{L}\left\{{ }_{a}^{C} D_{t}^{\alpha} f(t)\right\}(s)=s^{\alpha} \mathcal{L}\{f(t)\}-s^{\alpha-1} f(a), \quad 0<\alpha<1, \tag{2.15}
\end{equation*}
$$

where $\mathcal{L}$ is the Laplace transform operator. It is also assumed that f is a piecewise-continuous function of exponential order.

## Definition 6

[4] Let $f \in H^{1}(a, b), \alpha \in(0,1)$, then the Caputo-Fabrizio (CF) derivative of a function $f$ of order $\alpha$ is given by:

$$
\begin{equation*}
{ }_{a}^{C F} D_{t}^{\alpha} f(t)=\frac{(2-\alpha) \mathcal{F}(\alpha)}{2(1-\alpha)} \int_{a}^{t} f^{\prime}(\tau) \exp \left[-\frac{\alpha(t-\tau)}{1-\alpha}\right] d \tau, \tag{2.16}
\end{equation*}
$$

where $\mathcal{F}(\alpha)=(1-\alpha)+\frac{\alpha}{\Gamma(\alpha)}$, denotes a normalization function satisfying $\mathcal{F}(0)=\mathcal{F}(1)=1$.

## Theorem 7

[4] The Caputo-Fabrizio fractional integral operator of order $\alpha$ is given by

$$
\begin{equation*}
{ }_{a}^{C F} I_{t}^{\alpha} f(t)=\frac{2(1-\alpha)}{(2-\alpha) \mathcal{F}(\alpha)} f(t)+\frac{2 \alpha}{(2-\alpha) \mathcal{F}(\alpha)} \int_{a}^{t} f(\tau) d \tau \tag{2.17}
\end{equation*}
$$

## Definition 8

[4] The Laplace transform of the Caputo-Fabrizio derivative is given by

$$
\begin{equation*}
\mathcal{L}\left\{{ }_{a}^{C F} D_{t}^{\alpha} f(t)\right\}(s)=\frac{(2-\alpha) \mathcal{F}(\alpha)}{2} \frac{s \mathcal{L}\{f(t)\}-f(a)}{s+\alpha(1-s)} . \tag{2.18}
\end{equation*}
$$

## Definition 9

[5] Let $f \in H^{1}(a, b), \alpha \in(0,1)$, then the Atangana-Baleanu fractional derivative for a given function of order $\alpha$ in Caputo sense is defined by

$$
\begin{equation*}
{ }_{a}^{A B C} D_{t}^{\alpha} f(t)=\frac{\mathcal{F}(\alpha)}{(1-\alpha)} \int_{a}^{t} f^{\prime}(\tau) E_{\alpha}\left[-\alpha \frac{(t-\tau)^{\alpha}}{1-\alpha}\right] d \tau \tag{2.19}
\end{equation*}
$$

where $\mathcal{F}(\alpha)$, satisfying $\mathcal{F}(0)=\mathcal{F}(1)=1$, is a normalization function and $E_{\alpha}($.$) is the$ one-parameter Mittag-Leffler function, defined by,

$$
\begin{equation*}
E_{\alpha}(t)=\sum_{k=0}^{\infty} \frac{t^{k}}{\Gamma(\alpha k+1)}, \quad \alpha>0 \tag{2.20}
\end{equation*}
$$

Also, for $\alpha=1$, this becomes $E_{1}(t)=\sum_{k=0}^{\infty} \frac{t}{\Gamma(k+1)}=\sum_{k=0}^{\infty} \frac{t^{k}}{k!}=e^{t}$.

## Definition 10

[5] Atangana-Baleanu fractional integral of order $\alpha$ is defined as

$$
\begin{equation*}
{ }_{a}^{A B} I_{t}^{\alpha} f(t)=\frac{1-\alpha}{\mathcal{F}(\alpha)} f(t)+\frac{\alpha}{\mathcal{F}(\alpha) \Gamma(\alpha)} \int_{a}^{t} f(\tau)(t-\tau)^{\alpha-1} d \tau \tag{2.21}
\end{equation*}
$$

## Definition 11

[5] The Laplace transform for the Atangana-Baleanu fractional operator of order $\alpha$, where $0<\alpha<1$ is given as

$$
\begin{equation*}
\mathcal{L}\left\{{ }_{a}^{A B C} D^{\alpha} f(t)\right\}(s)=\frac{\mathcal{F}(\alpha)}{1-\alpha} \frac{s^{\alpha} \mathcal{L}\{f(t)\}-s^{\alpha-1} f(a)}{s^{\alpha}+\frac{\alpha}{1-\alpha}} \tag{2.22}
\end{equation*}
$$

## Basic idea of the NSFDs

We shall briefly discuss the basic idea of NSFDs and provide a general method for calculating denominator functions.
The first equation to be discretized is the decay equation given below

$$
\begin{equation*}
\frac{d x}{d t}=-\lambda x \tag{3.1}
\end{equation*}
$$

A simple discretization is to use a standard forward-Euler representation, i.e.,

$$
\begin{equation*}
\frac{x_{k+1}-x_{k}}{h}=-\lambda x_{k} \tag{3.2}
\end{equation*}
$$

This expression can be rewritten in the form

$$
\begin{equation*}
x_{k+1}=(1-\lambda h) x_{k} \tag{3.3}
\end{equation*}
$$

It can be observed that Eq. (3.3) leads to instabilities (or negative solutions, even with positive initial conditions) if $\lambda h>1$. However, if we use the fact that

$$
\begin{equation*}
1-\lambda h=e^{-\lambda h}+O\left(\lambda^{2} h^{2}\right) \tag{3.4}
\end{equation*}
$$

then for sufficiently small $\lambda h$, i.e., $0<\lambda . h \ll 1$, we can make the replacements

$$
1-\lambda h \rightarrow e^{-\lambda h}
$$

or

$$
h \rightarrow \frac{1-e^{-\lambda h}}{\lambda}
$$

Let us now select the denominator function for the discretization of the decay equation to be $\phi(h, \lambda)=\frac{1-e^{-\lambda h}}{\lambda}$, and, as a consequence, rewrite Eq. (3.3) in the form

$$
\begin{equation*}
\frac{x_{k+1}-x_{k}}{\left(\frac{1-e^{-\lambda h}}{\lambda}\right)}=-\lambda x_{k} \tag{3.5}
\end{equation*}
$$

which can be re-written as:

$$
\begin{equation*}
x_{k+1}=e^{-\lambda h} x_{k} \tag{3.6}
\end{equation*}
$$

It turns out that this is the exact finite difference scheme for Eq. (3.1). It is interesting to note that, Eq. (3.5), can also be derived by using an implicit forward-Euler scheme for Eq. (3.1), i.e.,

$$
\begin{equation*}
\frac{x_{k+1}-x_{k}}{h}=-\lambda x_{k+1} \tag{3.7}
\end{equation*}
$$

or

$$
\begin{equation*}
x_{k+1}=\frac{x_{k}}{(1+h \lambda)} \tag{3.8}
\end{equation*}
$$

Again, note that for $\lambda h \ll 1$, we have

$$
\begin{equation*}
1+\lambda h=e^{\lambda h}+O\left(\lambda^{2} h^{2}\right) \tag{3.9}
\end{equation*}
$$

or

$$
\begin{equation*}
x_{k+1}=\frac{x_{k}}{(1+h \lambda)} \tag{3.10}
\end{equation*}
$$

Again, note that for $\lambda h \ll 1$, we have

$$
\begin{equation*}
1+\lambda h=e^{\lambda h}+O\left(\lambda^{2} h^{2}\right) \tag{3.11}
\end{equation*}
$$

Using the argument made above, it follows that

$$
\begin{equation*}
h \rightarrow \frac{e^{\lambda h}-1}{\lambda} \tag{3.12}
\end{equation*}
$$

and the discretization is

$$
\begin{equation*}
\frac{x_{k+1}-x_{k}}{\left(\frac{e^{h \lambda}-1}{\lambda}\right)}=-\lambda x_{k+1} \tag{3.13}
\end{equation*}
$$

which again is the exact finite difference scheme for Eq. (3.1). Note also, that (3.13) can be rewritten as the following first-order difference equation

$$
\begin{equation*}
x_{k+1}=e^{-\lambda h} x_{k} \tag{3.14}
\end{equation*}
$$

Also, since they are exact finite difference schemes, these discretization hold for all $h>0$ and either sign for the parameter $\lambda$.

## Non- Standard Finite Difference Scheme for Caputo fractional derivative

The Caputo derivative of function $f(t)$ of order $\alpha \in(0,1)$ is defined as

$$
\begin{equation*}
{ }^{C} D_{t}^{\alpha}[f(t)]=\frac{1}{\Gamma(1-\alpha)} \int_{0}^{t} \frac{f^{\prime}(\theta)}{(t-\theta)^{\alpha}} d \theta \tag{3.15}
\end{equation*}
$$

The discretization of domain $[0, T$ ] is given as

$$
\begin{equation*}
t_{j}=j h, \quad j=0,1,2,3, \ldots \tag{3.16}
\end{equation*}
$$

where $h=\frac{T}{N}, N$ represent number of sub intervals and $T$ is final time. Now at $t=t_{j+1}$, Caputo derivative becomes

$$
\begin{equation*}
\left.{ }^{c} D_{t}^{\alpha}[f(t)]\right|_{t=t_{j+1}}=\frac{1}{\Gamma(1-\alpha)} \int_{0}^{t_{j+1}} \frac{f^{\prime}(\theta)}{\left(t_{j+1}-\theta\right)^{\alpha}} d \theta, \tag{3.17}
\end{equation*}
$$

or

$$
\begin{equation*}
\left.{ }^{c} D_{t}^{\alpha}[f(t)]\right|_{t=t_{j+1}}=\frac{1}{\Gamma(1-\alpha)} \sum_{k=0}^{j} \int_{t_{k}}^{t_{k+1}} f^{\prime}(\theta)\left(t_{j+1}-\theta\right)^{-\alpha} d \theta \tag{3.18}
\end{equation*}
$$

Now we approximate $f^{\prime}(\theta)=\frac{d f(\theta)}{d \theta}$ on $\left[t_{k}, t_{k+1}\right]$ as

$$
\begin{equation*}
\frac{d f(\theta)}{d \theta}=\frac{f^{k+1}-f^{k}}{\Psi(h)} \tag{3.19}
\end{equation*}
$$

where $f^{k}=f\left(t_{k}\right)$ and $\Psi(h)=\frac{e^{\mu h-1}}{\mu}=h+O\left(h^{2}\right)$.
Now (3.18) becomes

$$
\begin{equation*}
\left.{ }^{c} D_{t}^{\alpha}[f(t)]\right|_{t=t_{j+1}} \approx \frac{1}{\Gamma(1-\alpha)} \sum_{k=0}^{j} \int_{t_{k}}^{t_{k+1}} \frac{f^{k+1}-f^{k}}{\Psi(h)}\left(t_{j+1}-\theta\right)^{-\alpha} d \theta \tag{3.20}
\end{equation*}
$$

or

$$
\begin{equation*}
\left.{ }^{c} D_{t}^{\alpha}[f(t)]\right|_{t=t_{j+1}}=\frac{1}{\Gamma(2-\alpha)} \sum_{k=0}^{j} \frac{f^{k+1}-f^{k}}{\Psi(h)} A_{\alpha, j}^{k}, \tag{3.21}
\end{equation*}
$$

where

$$
\begin{equation*}
A_{\alpha, j}^{k}=(1-\alpha) \int_{t_{k}}^{t_{k+1}}\left(t_{j+1}-\theta\right)^{-\alpha} d \theta=h^{1-\alpha}\left[(j-k+1)^{1-\alpha}-(j-k)^{1-\alpha}\right] \tag{3.22}
\end{equation*}
$$

Consider the fractional order single population growth model:

$$
\begin{equation*}
{ }_{0}^{C} D_{t} N(t)=f(t, N(t))=r N(t), \quad N(0)=N_{0} \tag{3.23}
\end{equation*}
$$

Now the equation (3.23) can be written as

$$
\begin{equation*}
{ }^{c} D_{t}^{\alpha}[N(t)]=f(N(t)) . \tag{3.24}
\end{equation*}
$$

At $t=t_{j+1}$, we get

$$
\begin{equation*}
\left.{ }^{c} D_{t}^{\alpha}[N(t)]\right|_{t=t_{j+1}}=f\left(N\left(t_{j+1}\right)\right) \tag{3.25}
\end{equation*}
$$

Now using (3.21), we obtain that

$$
\begin{equation*}
\frac{1}{\Gamma(2-\alpha)} \sum_{k=0}^{j} \frac{N^{k+1}-N^{k}}{\Psi(h)} A_{\alpha, j}^{k}-f\left(N\left(t_{j+1}\right)\right)=0 \tag{3.26}
\end{equation*}
$$

Applying the scheme (3.26) to the fractional model (3.23), we have

$$
\begin{equation*}
\frac{1}{\Gamma(2-\alpha)} \sum_{k=0}^{j} \frac{N^{k+1}-N^{k}}{\Psi(h)} A_{\alpha, j}^{k}=r N^{j+1} \tag{3.27}
\end{equation*}
$$

or

$$
\begin{equation*}
\sum_{k=0}^{j}\left(N^{k+1}-N^{k}\right) A_{\alpha, j}^{k}=r N^{j+1} \Psi(h) \Gamma(2-\alpha) \tag{3.28}
\end{equation*}
$$

Note that, for $k=j, A_{\alpha, j}^{k}=h^{1-\alpha}$.

Thus,

$$
\begin{gather*}
\left(N^{j+1}-N^{j}\right) h^{1-\alpha}+\sum_{k=0}^{j-1}\left(N^{k+1}-N^{k}\right) A_{\alpha, j}^{k}=r N^{j+1} \Psi(h) \Gamma(2-\alpha)  \tag{3.29}\\
N^{j+1}=\frac{h^{1-\alpha} N^{j}-\sum_{k=0}^{j-1}\left(N^{k+1}-N^{k}\right) A_{\alpha, j}^{k}}{h^{1-\alpha}-r \Psi(h) \Gamma(2-\alpha)} \tag{3.30}
\end{gather*}
$$



## THANK YOU for listening!

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